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Defining the Nambu–Jona-Lasinio Model by Higher Derivative Kinetic Term

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ABSTRACT

Usual treatment of the Nambu–Jona-Lasinio (NJL) model using loop momentum cutoff suffers from ambiguities in choosing the loop momenta to be cut off and violation of (external) gauge invariance. We define the NJL model from the starting Lagrangian level by using a higher derivative fermion kinetic term with a cutoff parameter Λ. This definition is free from such ambiguities and manifestly keeps the chiral symmetry as well as the gauge invariance. Quantization of this higher derivative system, current operators and calculational method are discussed in some detail. Calculating the pion decay constant and $\pi^0 \to 2\gamma$ decay amplitude, we explicitly demonstrate that the low energy theorem holds. It is observed that the NJL mass relation $m_\sigma = 2m_0$ between the σ meson and fermion masses no longer holds in this model. We also present a simplified calculational method which is valid when the cutoff parameter Λ is much larger than the energy scale of physics.

1. Introduction

The Nambu–Jona-Lasinio (NJL) model[1,2] is not defined without an ultraviolet cutoff Λ. Usually this cutoff is introduced at the loop graph level. For instance, in the most popular treatment, the fermion one-loop integral is first rewritten using the Feynman parameter formula and the ultraviolet cutoff is made for the loop momentum variable with which the denominator of the integrand becomes an even function. Another treatment, which was adopted in the original NJL paper,[1] utilizes the dispersion relation. In either treatment, the cutoff is made graph by graph (or for each Green function separately). This implies that the theory is *not* defined at the starting Lagrangian level. This is very unsatisfactory.

Other serious problems in the usual treatment are the ambiguity in introducing the cutoff and the consistency with symmetries. The values of the divergent loop diagrams depend on the choice of the loop momentum variables for which we make the ultraviolet cutoff. We should specify the choice procedure unambiguously. Moreover the procedure has to be shown to be consistent with the chiral symmetry at least. To show this consistency would not be an easy task if the cutoff is introduced graph by graph. Furthermore if the system is coupled to external gauge fields the cutoff procedure should also be consistent with the gauge invariance. It is indeed very difficult to satisfy the gauge invariance if we adopt such a simple cutoff for the loop momentum variables.

A best example for the last problem is given by vector 2-point functions. Consider a NJL model which contains 4-fermi interaction in the vector channel and couples to the external photon field A_μ. If one rewrite the 4-fermi interaction term by introducing a vector auxiliary field V_μ, the fermion kinetic term gets to take the form

$$\bar{\psi} i\gamma^\mu(\partial_\mu - iV_\mu - ieA_\mu)\psi. \tag{1.1}$$

Note the pararellism between the vector field V_μ and the photon field A_μ. In calculating the fermion one-loop diagrams, there is no difference between them;

namely, exactly the same Feynman diagrams appear for the vector-vector, vector-photon and photon-photon 2-point functions. If we adopt a loop momentum cutoff to the diagram, we would obtain gauge non-invariant function not proportional to $g_{\mu\nu}p^2 - p^\mu p^\nu$, which may be a good result for the vector-vector function but is clearly unacceptable for the photon-photon and vector-photon functions. To achieve the gauge invariance for the latter, one sometimes throw away the gauge non-invariant piece, quadratic mass term $\propto g_{\mu\nu}\Lambda^2$, by hand. This might be a correct procedure. But if so, then, should we do the same also for the vector-vector case?

We propose in this paper to define the NJL model by using a higher derivative kinetic term for the fermion. We replace the usual fermion kinetic term by the following higher derivative one:

$$\bar{\psi} i \partial\!\!\!/ \psi \ \Rightarrow \ \bar{\psi} i \partial\!\!\!/ \big(1 + \frac{\partial\!\!\!/ \partial\!\!\!/}{\Lambda^2}\big)\psi \ . \tag{1.2}$$

Here the parameter Λ^2 plays the role of an ultraviolet cutoff. Note that this effective cutoff is made on each *fermion propagator* but not on each loop momentum, and that any loop diagrams are now well-defined and independent of the choice of the loop momenta. It should be emphasized that this defines the NJL model already at the starting *Lagrangian level*. It is manifest that this higher derivative definition keeps the important chiral symmetry of the system. Moreover if we switch on external gauge interactions (such as weak-electromagnetic ones), we can keep the gauge-invariance also by simply replacing each derivative factor $\partial\!\!\!/$ by a covariant one $D\!\!\!\!/$.

As a mere regularization method there is dimensional regularization or zeta function regularization which respects the gauge invariance. But what we need here is not a mere regularization but a definition of the NJL model by a 'regularization' which is not removed forever. If we define the NJL model by dimensional regularization, the model will be defined in $4-\epsilon$ dimensions with a certain ϵ fixed and then the physical meaning would become very unclear. In our definition by higher derivative, the cutoff parameter Λ has a physical meaning as the energy scale which gives an upper limit of the applicability of the model.

This paper is organized as follows. In Sect.2 we set up the NJL model which we discuss in this paper. Since the model is a higher derivative system rather different from the canonical one, we discuss there the problems of quantization, current operators and coupling to the external gauge fields in detail. In Sect.3 we present a method for reducing the calculations of Feynman diagrams in this system to those of the usual first-order derivative fermion system. We calculate some Green functions and note, in particular that the scalar meson has a pole below the two fermion threshold. In Sect.4 $\pi^0 \to 2\gamma$ amplitude is calculated and is explicitly demonstrated to be consistent with the low energy theorem. When the cutoff Λ is much larger than the energy scale we discuss, the calculation can be made much simpler. This is shown in Sect.5. Sect.6 is devoted to conclusion. In the Appendix, we present a generalized Noether procedure of constructing conserved current in a generic higher derivative system.

2. NJL Model Defined by Higher Derivative Kinetic Term

2.1. NJL Model and QCD-analogue Model

We consider a fermion $\psi = (\psi_{if})$ which carries $SU(N_\mathrm{c})$ color index i and $SU(n_\mathrm{f})$ flavor index f. The fundamental representation matrices are denoted by T^A $(A = 1, \cdots, N_\mathrm{c}^2 - 1)$ for the color $SU(N_\mathrm{c})$ and by λ^a $(a = 1, \cdots, n_\mathrm{f}^2 - 1)$ for the flavor $SU(n_\mathrm{f})$, respectively. They are normalized by $\mathrm{tr}(T^A T^B) = \frac{1}{2}\delta^{AB}$, $\mathrm{tr}(\lambda^a \lambda^b) = 2\delta^{ab}$. We also use flavor singlet matrix $\lambda^0 \equiv \sqrt{2/n_\mathrm{f}}\,\mathbf{1}_{n_\mathrm{f}}$ proportional to $n_\mathrm{f} \times n_\mathrm{f}$ unit matrix $\mathbf{1}_{n_\mathrm{f}}$. The NJL model we consider in this paper is the following system which possesses chiral $U(n_\mathrm{f})_\mathrm{R} \times U(n_\mathrm{f})_\mathrm{L}$ symmetry up to the axial $U(1)_\mathrm{A}$ anomaly:

$$
\begin{aligned}
\mathcal{L} = {}& \bar\psi i\partial\!\!\!/ \big(1 + \frac{\partial\!\!\!/\partial\!\!\!/}{\Lambda^2}\big)\psi \\
& + \frac{g_\mathrm{S}^2}{4\Lambda^2}\Big\{ \big[(\bar\psi\lambda^0\psi)^2 + (\bar\psi i\gamma_5\lambda^0\psi)^2\big] + \big[(\bar\psi\lambda^a\psi)^2 + (\bar\psi i\gamma_5\lambda^a\psi)^2\big] \Big\} \\
& - \frac{g_{\mathrm{V}0}^2}{8\Lambda^2}\big[(\bar\psi\gamma_\mu\lambda^0\psi)^2 + (\bar\psi\gamma_\mu\gamma_5\lambda^0\psi)^2\big] - \frac{g_\mathrm{V}^2}{8\Lambda^2}\big[(\bar\psi\gamma_\mu\lambda^a\psi)^2 + (\bar\psi\gamma_\mu\gamma_5\lambda^a\psi)^2\big] \ .
\end{aligned}
\tag{2.1}
$$

Here all fermion bilinears are of *color-singlet*. The fermion kinetic term is taken to be a higher derivative one as explained in the Introduction. This fully defines the model from the starting Lagrangian.

If we introduce auxiliary fields following the well-known procedure, this Lagrangian can equivalently be rewritten into

$$\begin{aligned}\mathcal{L} = \bar{\psi}\Big[i\partial\!\!\!/(1+\frac{\partial\!\!\!/\partial\!\!\!/}{\Lambda^2}) - \mathcal{M}\Big]\psi - \frac{\Lambda^2}{2g_{\mathrm{S}}^2}\,\mathrm{tr}\left(\Sigma^\dagger\Sigma\right) + \frac{\Lambda^2}{2g_{\mathrm{V}}^2}\,\mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right) \\ + \frac{\Lambda^2}{4}\left(\frac{1}{g_{\mathrm{V0}}^2} - \frac{1}{g_{\mathrm{V}}^2}\right)\left(\left[\mathrm{tr}(R_\mu\lambda^0)\right]^2 + \left[\mathrm{tr}(L_\mu\lambda^0)\right]^2\right) ,\end{aligned} \tag{2.2}$$

$$\mathcal{M}(x) \equiv \Sigma(x)\mathcal{P}_R + \Sigma^\dagger(x)\mathcal{P}_L - \gamma^\mu R_\mu(x)\mathcal{P}_R - \gamma^\mu L_\mu(x)\mathcal{P}_L \ .$$

with chiral projection operators $\mathcal{P}_{\mathrm{R}} \equiv (1+\gamma_5)/2$ and $\mathcal{P}_{\mathrm{L}} \equiv (1-\gamma_5)/2$. Eq.(2.2) gives equations of motion for the auxiliary fields as follows:

$$\begin{aligned}\Sigma &= -\frac{g_{\mathrm{S}}^2}{\Lambda^2}\left[\lambda^0(\bar{\psi}_{\mathrm{R}}\lambda^0\psi_{\mathrm{L}}) + \lambda^a(\bar{\psi}_{\mathrm{R}}\lambda^a\psi_{\mathrm{L}})\right] \\ R_\mu &= -\frac{g_{\mathrm{V}}^2}{2\Lambda^2}\left[\frac{g_{\mathrm{V0}}^2}{g_{\mathrm{V}}^2}\lambda^0(\bar{\psi}_{\mathrm{R}}\gamma_\mu\lambda^0\psi_{\mathrm{R}}) + \lambda^a(\bar{\psi}_{\mathrm{R}}\gamma_\mu\lambda^a\psi_{\mathrm{R}})\right] \\ L_\mu &= -\frac{g_{\mathrm{V}}^2}{2\Lambda^2}\left[\frac{g_{\mathrm{V0}}^2}{g_{\mathrm{V}}^2}\lambda^0(\bar{\psi}_{\mathrm{L}}\gamma_\mu\lambda^0\psi_{\mathrm{L}}) + \lambda^a(\bar{\psi}_{\mathrm{L}}\gamma_\mu\lambda^a\psi_{\mathrm{L}})\right]\end{aligned} \tag{2.3}$$

with $\psi_{\mathrm{R,L}} \equiv \mathcal{P}_{\mathrm{R,L}}\psi$. Note that all the auxiliary fields are $n_{\mathrm{f}} \times n_{\mathrm{f}}$ flavor matrices; Σ is a complex matrix while R^μ, L^μ are hermitian matrices.

The NJL model is often used as a model simulating QCD.[3] If we consider a single gluon exchange, it may effectively be expressed by the following four-fermion interaction:[4]

$$\mathcal{L}^{\mathrm{int}}_{\text{QCD-analogue}} = -\frac{g^2}{\Lambda^2}(\bar{\psi}\gamma_\mu T^A\psi)(\bar{\psi}\gamma^\mu T^A\psi) \ , \tag{2.4}$$

where Λ is a (suitable) characteristic energy scale of QCD and g is the color gauge coupling constant. If we perform the Fierz transformation and keep only the leading

terms in $1/N_{\rm c}$, this four-fermi interaction (2.4) can be rewritten into the same form as that in Eq.(2.1) with identification

$$g_{\rm S}^2 = g_{\rm V}^2 = g_{\rm V0}^2 = g^2 \ . \tag{2.5}$$

(If we also keep non-leading terms in $1/N_{\rm c}$, only the flavor-singlet vector four-fermi coupling is replaced by $g_{\rm V0}^2 = (1 - 2n_{\rm f}/N_{\rm c})\, g^2$.) We refer to this four-fermi interaction system with coupling relations (2.5) as 'QCD-analogue NJL model'.

2.2. Quantization

Let us now consider the quantization of this higher derivative system. The procedure has long been known since Pais and Uhlenbeck,[5] and we here follow the procedure by Nakanishi.[6] Consider generally a system which contains higher derivatives only in the kinetic term as follows:

$$\mathcal{L} = \bar\psi f(i\partial\!\!\!/)\psi + \mathcal{L}_{\rm int}(\psi, \bar\psi), \tag{2.6}$$

where $f(x)$ is a polynomial of the form

$$f(x) = a \prod_{j=0}^{n} (x - m_j) \qquad\qquad (m_j \neq m_k \ \text{for}\ j \neq k) \tag{2.7}$$

and the interaction part $\mathcal{L}_{\rm int}$ is assumed to contain no derivatives. The well-known partial fraction formula $1/f(x) = \sum_j [f'(m_j)(x - m_j)]^{-1}$ leads to an identity

$$\sum_{j=0}^{n} a\eta_j \Big[\prod_{k\neq j} (x - m_k) \Big] = 1 \qquad\qquad \Big(\eta_j = \frac{1}{f'(m_j)}\Big) \ . \tag{2.8}$$

Using this, we can decompose the fermion field ψ as

$$\psi = \sum_{j=0}^{n} \psi_j \qquad \text{where} \qquad \psi_j \equiv a\eta_j \Big[\prod_{k\neq j} (i\partial\!\!\!/ - m_k) \Big] \psi \ , \tag{2.9}$$

and then, by noting $f(i\partial\!\!\!/)\psi = \eta_j^{-1}(i\partial\!\!\!/ - m_j)\psi_j$ independently of j, the higher

derivative Lagrangian (2.6) is seen to be rewritten into

$$\mathcal{L} = \sum_{j=0}^{n} \eta_j^{-1} \bar{\psi}_j (i\partial\!\!\!/ - m_j)\psi_j + \mathcal{L}_{\mathrm{int}} \left(\psi = \sum\nolimits_j \psi_j,\ \bar{\psi} = \sum\nolimits_j \bar{\psi}_j\right) . \tag{2.10}$$

It is now clear in this form that the original higher derivative system is equivalent to a system consisting of ordinary (positive or negative metric *) fermion fields ψ_j with mass m_j, to which the usual canonical quantization procedure is applicable. Then clearly the free propagator of the j-th fermion is given by $i\eta_j/(p\!\!\!/ - m_j)$ and therefore that of the original field ψ is found to be given by

$$\begin{aligned}\mathrm{F.T.}\left\langle \mathrm{T}\psi\bar{\psi}\right\rangle &= \sum_{j=0}^{n} \mathrm{F.T.}\left\langle \mathrm{T}\psi_j\bar{\psi}_j\right\rangle \\ &= \sum_{j=0}^{n} \eta_j \frac{i}{p\!\!\!/ - m_j} = \frac{i}{a\prod_{j=0}^{n}(p\!\!\!/ - m_j)} = \frac{i}{f(p\!\!\!/)} .\end{aligned} \tag{2.11}$$

Namely, the naive expectation that the propagator of ψ is given by the inverse of the kinetic term is correct in this case. Conversely, if we take it for granted that the propagator of ψ is given by $i/f(p\!\!\!/)$, we can start from it and decompose the ψ-propagator into partial fractions $\sum_{j=0}^{n} i\eta_j/(p\!\!\!/ - m_j)$ in every Feynman diagram. Then it is easy to see that the theory is equivalent to the above system (2.10) consisting of $(n+1)$-fermions ψ_j with mass m_j.

We can apply this general procedure to our NJL system (2.2) in various ways depending on which part we regard as the free kinetic term $\mathcal{L}_{\mathrm{free}} \equiv \bar{\psi} f(i\partial\!\!\!/)\psi$. We now discuss two ways, separately.

A picture

The simplest way, which we call 'A picture', is to take the original kinetic term $\bar{\psi} i\partial\!\!\!/ \left(1 + \partial\!\!\!/\partial\!\!\!//\Lambda^2\right)\psi$ as $\mathcal{L}_{\mathrm{free}}$. Then $f(x) = x(1 - x/\Lambda)(1 + x/\Lambda)$ and it has three

$\star$ If we arrange the roots m_j in order of their values, the weights $\eta_j = 1/f'(m_j)$ take alternating signs and hence ψ_j become of positive and negative metric alternatingly.

roots $m_j = 0, \Lambda, -\Lambda$ with $\eta_j = 1, -1/2, -1/2$. Accordingly, the original fermion field ψ is decomposed into $\psi = \psi_0 + \psi_\Lambda + \psi_{-\Lambda}$ with $\psi_{0,\pm\Lambda}$ denoting the component fermions with masses $m_j = 0, \Lambda, -\Lambda$, respectively, and the free part lagrangian is written in the form:

$$\begin{aligned}\mathcal{L}^{(0)}_{\text{free}} &\equiv \bar{\psi} i\partial\!\!\!/ \bigl(1 + \frac{\partial\!\!\!/\partial\!\!\!/}{\Lambda^2}\bigr)\psi \\ &= \bar{\psi}_0 i\partial\!\!\!/ \psi_0 - 2\bar{\psi}_\Lambda (i\partial\!\!\!/ - \Lambda)\psi_\Lambda - 2\bar{\psi}_{-\Lambda}(i\partial\!\!\!/ + \Lambda)\psi_{-\Lambda} \ .\end{aligned} \tag{2.12}$$

This A picture hence corresponds to the following decomposition of the fermion propagator:

$$\frac{1}{-p\!\!\!/(1 - \frac{p^2}{\Lambda^2})} = \frac{1}{-p\!\!\!/} - \frac{1}{2}\frac{1}{\Lambda - p\!\!\!/} - \frac{1}{2}\frac{1}{-\Lambda - p\!\!\!/} \ . \tag{2.13}$$

The expression (2.9) of the component fermions $\psi_{0,\pm\Lambda}$ in terms of the original fermion ψ now explicitly reads

$$\begin{aligned}\psi_0 &= \Lambda^{-2}(\partial\!\!\!/\partial\!\!\!/ + \Lambda^2)\psi \\ \psi_{\pm\Lambda} &= \frac{1}{2\Lambda^2}(-\partial\!\!\!/\partial\!\!\!/ \pm i\partial\!\!\!/\Lambda)\psi \ .\end{aligned} \tag{2.14}$$

B picture

In some cases, it is more convenient to choose another form for the free kinetic term $\mathcal{L}_{\text{free}}$. Indeed when $\Sigma(x)$ develops a nonvanishing vacuum expectation value (VEV) $\langle 0|\,\Sigma(x)\,|0\rangle = m\mathbf{1}_{n_{\text{f}}} \neq 0$, the fermion acquires a mass term $-m\bar{\psi}\psi$. In such a case, we can take the following lagrangian as the free kinetic term by including the mass term:

$$\mathcal{L}^{(m)}_{\text{free}} = \bar{\psi} i\partial\!\!\!/ (1 + \frac{\partial\!\!\!/\partial\!\!\!/}{\Lambda^2})\psi - m\bar{\psi}\psi \ . \tag{2.15}$$

Then, for this choice, we have

$$f(x) = x(1 - x^2/\Lambda^2) - m \equiv -\Lambda^{-2}(x - m_0)(x - m_1)(x - m_2) \ , \tag{2.16}$$

and the kinetic term (2.15) is rewritten into

$$\mathcal{L}_{\rm free}^{(m)} = \eta_0^{-1}\bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 + \eta_1^{-1}\bar{\psi}_1(i\partial\!\!\!/ - m_1)\psi_1 + \eta_2^{-1}\bar{\psi}_2(i\partial\!\!\!/ - m_2)\psi_2\ . \qquad (2.17)$$

We refer to this as 'B picture', which corresponds to the following decomposition of the fermion propagator:

$$\frac{1}{m - p\!\!\!/(1-\frac{p^2}{\Lambda^2})} = \eta_0\frac{1}{m_0 - p\!\!\!/} + \eta_1\frac{1}{m_1 - p\!\!\!/} + \eta_2\frac{1}{m_2 - p\!\!\!/}\ . \qquad (2.18)$$

An inconvenience for analytic treatment in this choice is that we have no simple explicit expressions for the three masses m_j (and hence the weights $\eta_j = [f'(m_j)]^{-1}$ also); they are determined by Eq.(2.16) and their explicit expressions can only be given by a complicated Cardano's formula. However, for the purpose to do practical calculations, we can make use of computer. Then such complication is of no problem and the decomposition (2.17) reducing the higher derivative system into the usual fermion system provides us with very efficient tool to calculate various quantities, as we shall see explicitly in Sect.3.

We assume henceforth that we are labeling the three masses (m_0, m_1, m_2) in order such that they approach to $(0, +\Lambda, -\Lambda)$, respectively, as the mass parameter m goes to zero. So ψ_0 with mass m_0 is the physical fermion with positive metric and the other $\psi_{1,2}$ are unphysical fermions with negative metric. We should note the fact that possible value of the physical fermion mass m_0 is bounded from above by

$$m_0 \leq \frac{\Lambda}{\sqrt{3}} = 0.57735\cdots\Lambda, \qquad (2.19)$$

although the mass parameter m can be arbitrarily large in principle. This is because the root m_0 becomes complex beyond this limit. Indeed, the cubic polynomial $f(x)$ in (2.16) has two stationary points at $\pm\Lambda/\sqrt{3}$ and the root m_0, if being real, has to lie in between them. This bound (2.19) is not a defect of the present definition of the NJL model but is of physical significance. The NJL model is defined with

an ultraviolet cutoff in any case and can only describe physics below it. Therefore the physical fermion mass m_0 generated by the NJL model dynamics can be at most of the same order as the cutoff and otherwise becomes unreliable.

The expression (2.9) of the component fermions ψ_j in terms of the original one ψ now reads

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix} = -\Lambda^{-2} \begin{pmatrix} \eta_0 & \eta_0 m_0 & -\eta_0 \Lambda^2 \frac{m}{m_0} \\ \eta_1 & \eta_1 m_1 & -\eta_1 \Lambda^2 \frac{m}{m_1} \\ \eta_2 & \eta_2 m_2 & -\eta_2 \Lambda^2 \frac{m}{m_2} \end{pmatrix} \begin{pmatrix} -\Box\psi \\ i\partial\!\!\!/\psi \\ \psi \end{pmatrix} , \tag{2.20}$$

where we have used the following relations for the three masses m_j which directly follow from the defining equation (2.16):

$$m_0 + m_1 + m_2 = 0, \qquad m_0 m_1 + m_1 m_2 + m_2 m_0 = -\Lambda^2, \qquad m_0 m_1 m_2 = -m\Lambda^2 \ . \tag{2.21}$$

For later convenience, we here cite some formulas for the masses m_j and the weights η_j. Using the relations (2.21) and $m_j^3 = \Lambda^2(m_j - m)$ for $\forall j$, one can easily derive

$$\begin{gathered} \sum_{j=0}^{2} m_j^2 = 2\Lambda^2, \qquad \sum_{j=0}^{2} m_j^3 = -3m\Lambda^2, \qquad \sum_{j=0}^{2} m_j^4 = 2\Lambda^4, \\ \sum_{i>j} m_i^2 m_j^2 = \Lambda^4, \qquad \frac{m}{m_j} = 1 - \frac{m_j^2}{\Lambda^2}, \qquad etc.. \end{gathered} \tag{2.22}$$

The defining equation of the weights η_j together with these mass relations lead to the following relations:

$$\begin{gathered} \eta_j = \left(\frac{m}{m_j} - 2\frac{m_j^2}{\Lambda^2}\right)^{-1} = \left(3\frac{m}{m_j} - 2\right)^{-1}, \qquad \sum_{j=0}^{2} \eta_j = 0, \quad \sum_{j=0}^{2} \eta_j m_j = 0, \\ \sum_{j=0}^{2} \eta_j m_j^2 = -\Lambda^2, \quad \sum_{j=0}^{2} \eta_j m_j^3 = 0, \quad \sum_{j=0}^{2} \frac{\eta_j}{m_j} = \frac{1}{m}, \quad \sum_{j=0}^{2} \frac{\eta_j}{m_j^2} = \frac{1}{m^2}, \qquad etc.. \end{gathered} \tag{2.23}$$

2.3. Current Operators

We have emphasized that our definition of NJL model respects the chiral symmetry. It is indeed clear that our Lagrangian (2.2) is invariant under the chiral $U(n_{\rm f})_{\rm R} \times U(n_{\rm f})_{\rm L}$ transformation. Let us derive current operators corresponding to this symmetry since they take different forms from the usual ones because of the presence of higher derivatives.

To do this, the simplest way is to use the A picture. Then the Lagrangian is given by (2.12) (plus interaction term containing no derivatives) to which the usual Noether procedure applies. In view of the expression (2.14) for the component fields $\psi_{0,\pm\Lambda}$ in terms of the original ψ, we see that the vector transformation $\psi \to \psi' = e^{i\theta}\psi$ ($\theta \equiv \sum_a \theta^a \lambda^a$) on the original fermion field ψ is realized on the component fields $\psi_{0,\pm\Lambda}$ in the same form, $\psi_{0,\pm\Lambda} \to \psi'_{0,\pm\Lambda} = e^{i\theta}\psi_{0,\pm\Lambda}$, while the chiral transformation $\psi \to \psi' = e^{i\gamma_5\theta}\psi$ is realized on the component fields a bit differently as follows:

$$
\begin{aligned}
{\psi'}_0 &= e^{i\gamma_5\theta}\psi_0 \\
{\psi'}_\Lambda + {\psi'}_{-\Lambda} &= e^{i\gamma_5\theta}(\psi_\Lambda + \psi_{-\Lambda}) \\
{\psi'}_\Lambda - {\psi'}_{-\Lambda} &= e^{-i\gamma_5\theta}(\psi_\Lambda - \psi_{-\Lambda}) \ .
\end{aligned}
\tag{2.24}
$$

Having found the transformation law of the component fields $\psi_{0,\pm\Lambda}$, we can apply the usual Noether procedure and obtain the corresponding currents. The vector current is found to be

$$
j^\mu_a = \bar\psi_0 \lambda^a \gamma^\mu \psi_0 - 2\bar\psi_\Lambda \lambda^a \gamma^\mu \psi_\Lambda - 2\bar\psi_{-\Lambda} \lambda^a \gamma^\mu \psi_{-\Lambda} \ , \tag{2.25}
$$

and the axial current is given by

$$
j^\mu_{5a} = \bar\psi_0 \lambda^a \gamma^\mu \gamma_5 \psi_0 - 2\bar\psi_\Lambda \lambda^a \gamma^\mu \gamma_5 \psi_{-\Lambda} - 2\bar\psi_{-\Lambda} \lambda^a \gamma^\mu \gamma_5 \psi_\Lambda \ . \tag{2.26}
$$

We can rewrite these current expressions in terms of the original fermion field ψ

by using relations (2.14) and find

$$
\begin{aligned}
j_a^\mu &= \bar{\psi}\lambda^a\gamma^\mu\psi + \frac{1}{\Lambda^2}\big[\Box\bar{\psi}\cdot\lambda^a\gamma^\mu\psi - \bar{\psi}\overleftarrow{\partial\!\!\!/}\,\lambda^a\gamma^\mu\partial\!\!\!/\psi + \bar{\psi}\lambda^a\gamma^\mu\Box\psi_{-\Lambda}\big] \ , \\
j_{5a}^\mu &= \bar{\psi}\lambda^a\gamma^\mu\gamma_5\psi + \frac{1}{\Lambda^2}\big[\Box\bar{\psi}\cdot\lambda^a\gamma^\mu\gamma_5\psi + \bar{\psi}\overleftarrow{\partial\!\!\!/}\,\lambda^a\gamma^\mu\gamma_5\partial\!\!\!/\psi + \bar{\psi}\lambda^a\gamma^\mu\gamma_5\Box\psi_{-\Lambda}\big] \ .
\end{aligned}
\tag{2.27}
$$

Although not being well-known, there is in fact a direct and general procedure for deriving the Noether current (without decomposing ψ) for generic higher derivative systems, which we show in the Appendix. We can see that the Noether current obtained by the direct procedure coincides with the above (2.27) up to so-called ambiguity term of the form $\partial_\nu f^{\mu\nu}$ with an antisymmetric tensor $f^{\mu\nu}$.

In some cases we need current expression written in terms of the component fields ψ_j in B picture. This can be obtained if we invert the relation (2.20) between ψ_j and $(\psi, \partial\!\!\!/\psi, \Box\psi)$:

$$
\begin{pmatrix} -\Box\psi \\ i\partial\!\!\!/\psi \\ \psi \end{pmatrix} = \begin{pmatrix} m_0^2 & m_1^2 & m_2^2 \\ m_0 & m_1 & m_2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix} \ . \tag{2.28}
$$

Substituting this into (2.27), we find the currents in the B picture:

$$
\begin{aligned}
j_a^\mu &= \sum_{j=0}^{2} \eta_j^{-1}\bar{\psi}_j\lambda^a\gamma^\mu\psi_j \ , \\
j_{5a}^\mu &= \sum_{j=0}^{2} \eta_j^{-1}\bar{\psi}_j\lambda^a\gamma^\mu\gamma_5\psi_j + \sum_{i,j=0}^{2}\frac{2m_im_j}{\Lambda^2}\bar{\psi}_i\lambda^a\gamma^\mu\gamma_5\psi_j \ .
\end{aligned}
\tag{2.29}
$$

2.4. Coupling to External Gauge Fields

It is easy to couple external gauge fields to the system by gauging the chiral $U(n_\mathrm{f})_\mathrm{R} \times U(n_\mathrm{f})_\mathrm{L}$ symmetry. This is achieved simply by replacing the kinetic term by the covariant derivative one:

$$
\begin{aligned}
&\bar\psi i\not\partial\big(1+\frac{\not\partial\not\partial}{\Lambda^2}\big)\psi \\
&\quad\longrightarrow\ \bar\psi i\not D\big(1+\frac{\not D\not D}{\Lambda^2}\big)\psi = \bar\psi i(\not\partial - i\not\mathcal{V} - i\not\mathcal{A}\gamma_5)\Big(1+\frac{(\not\partial - i\not\mathcal{V} + i\not\mathcal{A}\gamma_5)(\not\partial - i\not\mathcal{V} - i\not\mathcal{A}\gamma_5)}{\Lambda^2}\Big)\psi\ ,
\end{aligned}
\tag{2.30}
$$

where $\mathcal{V}_\mu = \sum_a \mathcal{V}_\mu^a \lambda^a$ and $\mathcal{A}_\mu = \sum_a \mathcal{A}_\mu^a \lambda^a$ are the vector and axial-vector gauge fields of the chiral $U(n_\mathrm{f})_\mathrm{R} \times U(n_\mathrm{f})_\mathrm{L}$ group. If we gauge a part of the group, the gauge fields should also be understood so; for instance, if we couple only the photon A_μ to the system, then $\mathcal{V}_\mu = eQA_\mu$ and $\mathcal{A}_\mu = 0$ with Q being the charge quantum number matrix of the fermion. Note that the covariant derivative D_μ of course depends on the operand and $D_\mu\psi = (\partial_\mu - i\mathcal{V}_\mu - i\mathcal{A}_\mu\gamma_5)\psi$ while $D_\mu\not D\psi = (\partial_\mu - i\mathcal{V}_\mu + i\mathcal{A}_\mu\gamma_5)\not D\psi$.

Here two remarks are in order. First we note that the above covariant kinetic term contains the terms linear in the gauge fields in the form,

$$
\mathcal{V}_\mu^a j_a^\mu + \mathcal{A}_\mu^a j_{5a}^\mu + (\text{total derivatives}),
$$

and the currents j_a^μ and j_{5a}^μ just coincide with the above ones given in (2.27). Second, in the above covariantization we started with the expression $\bar\psi i\not\partial\big(1 + (\not\partial\not\partial/\Lambda^2)\big)\psi$. But if we started with an equivalent one $\bar\psi i\not\partial\big(1 + (\Box/\Lambda^2)\big)\psi$, we would have obtained $\bar\psi i\not D\big(1 + (D^2/\Lambda^2)\big)\psi$, which is different from the above one $\bar\psi i\not D\big(1 + (\not D\not D/\Lambda^2)\big)\psi$. The difference is, however, seen to be a non-minimal interaction term (Pauli term) of the form like $F_{\mu\nu}\bar\psi\gamma^\mu\gamma^\nu\psi$ with a field strength $F_{\mu\nu}$, and hence the currents defined by the linear terms in the gauge fields again coincides with the above ones up to ambiguity terms of the form $\partial_\nu f^{\mu\nu}$.

We have seen that our definition of the NJL model by higher derivative is made consistent with the gauge symmetry also. However, we should note the

fact that the vertex functions of the external gauge fields alone are not yet well-regularized by our higher derivative fermion kinetic term. Consider generally a fermion one-loop diagram which contains n vertices of 'mesons' Σ, R_μ and L_μ and m vertices of external gauge fields $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$. The fermion propagator behaves as $\sim k^{-3}$ at high loop momentum k and the vertex factor for 'mesons' contains no momentum. But the point is that the vertex of the external gauge fields contains second powers of fermion momenta as the currents (2.27) show. Therefore the diagram has superficial degree of divergence

$$\omega = 4 - 3(n+m) + 2m = 4 - 3n - m.$$

This is non-negative when $(n = 1, m = 0, 1)$ and $(n = 0, m = 1, 2, 3, 4)$ aside from an irrelevant vacuum graph case $(n = m = 0)$. The linear or logarithmic divergences for the former cases $(n = 1, m = 0, 1)$ are in fact absent because of Lorentz, chiral and gauge invariances. Indeed, first, the $(n = 1, m = 0)$ case corresponds to 'meson' tadpoles (1-point functions), and Σ-tadpole vanishes by the chiral symmetry while vector R_μ and L_μ tadpoles vanish by Lorentz invariance. Second, the $(n = 1, m = 1)$ case corresponds to 'meson'-gauge transition 2-point vertex functions; when the meson is Σ, the 2-point vertex Σ-$\mathcal{V}_\mu$ or Σ-$\mathcal{A}_\mu$ vanishes by chiral symmetry, and when the meson is R_ν or L_ν, the 2-point vertex like $(R + L)_\nu$-$\mathcal{V}_\mu$ should be proportional to $(g_{\mu\nu}p^2 - p_\mu p_\nu)$ by gauge invariance. Therefore they, being superficially logarithmically divergent, become convergent actually. The divergences thus occur only when $(n = 0, m = 1, 2, 3, 4)$, *i.e.*, for the *diagrams consisting of external gauge fields alone*. For those diagrams the present higher derivative kinetic term does not improve the divergence situation at all; however higher the fermion kinetic term derivatives is chosen, the gauge boson vertex also gets to contain higher derivatives accordingly. This is a well-known fact in the higher derivative regularization.[7] We can however regularize those divergence using dimensional or Pauli-Villars or any other suitable regularization.⋆

⋆ Note that this is a mere regularization which is to be removed eventually by the usual renormalization.

For definiteness, we adopt dimensional regularization in this paper.

3. Example Calculations

3.1. Effective Action and Potential

As we have seen in the above, the naive Feynman rule is correct for this type of higher derivative system. This also implies that the usual Feynman path integral expression for the Green function generating functional is valid for our system with lagrangian $\mathcal{L}$ given by (2.2). Therefore the effective action Γ for the 'meson' fields $\Sigma, \Sigma^\dagger, R_\mu$ and L_μ is given in the leading order in $1/N_{\rm c}$ simply by integrating over the fermion field ψ (and $\bar\psi$):

$$
\begin{aligned}
\Gamma = \int d^4x\, [-\frac{\Lambda^2}{2g_{\rm S}^2}\,{\rm tr}\left(\Sigma^\dagger\Sigma\right) + \frac{\Lambda^2}{2g_{\rm V}^2}\,{\rm tr}\left(R^\mu R_\mu + L^\mu L_\mu\right)] \\
+ \frac{N_{\rm c}}{i}{\rm TrLn}[i\partial\!\!\!/(1+\frac{\partial\!\!\!/\partial\!\!\!/}{\Lambda^2}) - \mathcal{M}]\ .
\end{aligned}
\tag{3.1}
$$

[Here we have omitted the last term in (2.2), $(\Lambda^2/4)\left(g_{\rm V0}^{-2} - g_{\rm V}^{-2}\right)\left(\left[{\rm tr}(R_\mu\lambda^0)\right]^2 + \left[{\rm tr}(L_\mu\lambda^0)\right]^2\right)$ simply for brevity of writing. Namely we may understand that we consider only the $g_{\rm V} = g_{\rm V0}$ case henceforth, or otherwise, it should be understood that this vector singlet term is always accompanying the vector term $(\Lambda^2/2)\,{\rm tr}(R^\mu R_\mu + L^\mu L_\mu)$.] Precisely speaking, the second TrLn term standing for the fermion one-loop diagrams is not made convergent yet by our third order derivative propagator. But if we expand it with respect to $\mathcal{M}$, the ultraviolet divergence appears only in the first two terms; the zero-th and first order terms in $\mathcal{M}$. Since the first order term vanishes by chiral symmetry and Lorentz invariance as we explained above, the divergence in fact appears only in the zero-th order term which is a field independent constant we can throw away freely.

The effective potential $V(\Sigma)$, whose stationary point determines the VEV of

Σ, is easily found from (3.1) to be given by

$$
\begin{aligned}
V(\Sigma) &= \frac{\Lambda^2}{2g_{\mathrm{S}}^2}\,\mathrm{tr}(\Sigma^\dagger\Sigma) - N_{\mathrm{c}} \int \frac{d^4p}{i(2\pi)^4}\,\mathrm{tr}\ln\left[\Sigma\mathcal{P}_{\mathrm{R}} + \Sigma^\dagger\mathcal{P}_{\mathrm{L}} - \not{p}\big(1 - \frac{\not{p}\not{p}}{\Lambda^2}\big)\right] \\
&= \frac{\Lambda^2}{2g_{\mathrm{S}}^2}\,\mathrm{tr}(\Sigma^\dagger\Sigma) - 2N_{\mathrm{c}} \int \frac{d^4p}{i(2\pi)^4}\,\mathrm{tr}\ln\left[\Sigma^\dagger\Sigma - p^2\big(1 - \frac{p^2}{\Lambda^2}\big)^2\right] ,
\end{aligned}
\tag{3.2}
$$

where tr in the second integral expression denotes the trace only over the flavor space. The scalar field Σ is generally expanded into

$$
\Sigma = \sigma + i\pi\ , \qquad \sigma = \sum_{a=0}^{n_{\mathrm{f}}^2-1} \sigma^a \frac{\lambda^a}{2}\ , \qquad \pi = \sum_{a=0}^{n_{\mathrm{f}}^2-1} \pi^a \frac{\lambda^a}{2}\ , \tag{3.3}
$$

and we see that the VEV of Σ is always brought into a real and diagonal matrix form by the chiral $U(n_{\mathrm{f}})_{\mathrm{R}} \times U(n_{\mathrm{f}})_{\mathrm{L}}$ rotation and further that the diagonal matrix is in fact proportional to unit matrix since each of the diagonal values is separately determined by minimizing exactly the same form of potential. Therefore, without of loss of generality, we can substitute

$$
\langle\Sigma\rangle = \sigma^0\lambda^0 \equiv m\mathbf{1}_{n_{\mathrm{f}}} \tag{3.4}
$$

into the effective potential, where m is a mass parameter which will yield a fermion mass term $-m\bar{\psi}\psi$. Then we find the effective potential to be

$$
\frac{1}{n_{\mathrm{f}}} V\big(\Sigma = m\mathbf{1}_{n_{\mathrm{f}}}\big) = \frac{\Lambda^2}{2g_{\mathrm{S}}^2} m^2 - 2N_{\mathrm{c}} \int \frac{d^4p}{i(2\pi)^4} \ln\left[m^2 - p^2\big(1 - \frac{p^2}{\Lambda^2}\big)^2\right] . \tag{3.5}
$$

The stationary condition of the potential leads to the following gap equation which determines the non-zero mass value m corresponding to spontaneous chiral symmetry breaking:

$$
\frac{\Lambda^2}{4g_{\mathrm{S}}^2 N_{\mathrm{c}}} - \int \frac{d^4p}{i(2\pi)^4} \frac{1}{m^2 - p^2\big(1 - \frac{p^2}{\Lambda^2}\big)^2} = 0\ . \tag{3.6}
$$

Up to here the story is formally the same as in the usual NJL model case and the difference is only in the higher derivative term in the fermion propagator. As

noted in the previous section, the higher derivative propagator can be reduced to the sum of ordinary fermion propagators and then we can evaluate the quantities in the same manner as usual. For instance, consider the effective potential in (3.5). If we use Eq.(2.16) with $x = \pm\not{p}$ substituted, we can evaluate it as

$$\frac{1}{n_{\rm f}} V\big(\Sigma = m\mathbf{1}_{n_{\rm f}}\big) = \frac{\Lambda^2}{2g_{\rm S}^2} m^2 + N_{\rm c} \sum_{j=0}^{2} F_0(m_j) \ , \tag{3.7}$$

where m_j are the three masses of the component fermion ψ_j, determined by Eq.(2.16), and the function $F_0(m_j)$ is the usual fermion vacuum energy given by

$$\begin{aligned} F_0(m_j) &= -\int \frac{d^n p}{i(2\pi)^n} \operatorname{tr} \ln(m_j - \not{p}) = -2 \int \frac{d^n p}{i(2\pi)^n} \ln(m_j^2 - p^2) \\ &= -\frac{1}{16\pi^2} m_j^4 \left(\mathbf{L} + \ln \frac{m_j^2}{\Lambda^2} - \frac{3}{2} \right) + O(\epsilon) \ , \end{aligned} \tag{3.8}$$

with $\mathbf{L}$ standing for the 'divergent' part:

$$\begin{aligned} &\mathbf{L} \equiv \ln \Lambda^2 - \frac{1}{\bar{\epsilon}}, \\ &\frac{1}{\bar{\epsilon}} \equiv \frac{1}{\epsilon} - \gamma + \ln(4\pi), \qquad \epsilon \equiv \frac{4-n}{2}, \qquad \gamma: \ \text{Euler constant} \ . \end{aligned} \tag{3.9}$$

Here note that we have used dimensional regularization to evaluate the contributions from the component fermions. This is always necessary since those contributions are separately ultraviolet-divergent, although the sum is generally convergent. In this case of effective potential, however, the sum still contains an m-independent divergence as mentioned above. In fact, the sum $\sum_{j=0}^{2} F(m_j)$ contains the divergent part $-(\mathbf{L}/16\pi^2) \sum_j m_j^4$ and it is seen to be m-independent constant because of an equality $\sum_j m_j^4 = 2\Lambda^4$ in (2.22). Discarding the m-independent divergence and constants, we find the potential (3.7) to yield

$$\frac{1}{n_{\rm f}} V\big(\Sigma = m\mathbf{1}_{n_{\rm f}}\big) = \frac{\Lambda^2}{2g_{\rm S}^2} m^2 - \frac{N_{\rm c}}{16\pi^2} \sum_{j=0}^{2} m_j^4 \ln \frac{m_j^2}{\Lambda^2} \ . \tag{3.10}$$

The constant is adjusted so that this becomes zero as $m \to 0$ in which (m_0, m_1, m_2) $\to (0, +\Lambda, -\Lambda)$. The gap equation (3.6) can also be rewritten as follows if we use

an equality $\left[m^2 - p^2\left(1-(p^2/\Lambda^2)\right)^2\right]^{-1} = \sum_j \eta_j (m_j/m)[m_j^2 - p^2]^{-1}$ which follows from taking trace of both sides of Eq.(2.18):

$$\frac{\Lambda^2}{4g_{\mathrm{S}}^2 N_{\mathrm{c}}} = \frac{1}{16\pi^2}\sum_{j=0}^{2}\eta_j \frac{m_j^3}{m}\ln\frac{m_j^2}{\Lambda^2}\ . \tag{3.11}$$

3.2. Two-point Functions

The same technique applies also to other quantities. For instance, the n-point Green functions on the spontaneously broken vacuum is obtained by expanding the effective action (3.1) around $\mathcal{M} = m$. So the fermion propagator is given by $i[\not{p}(1-p^2/\Lambda^2) - m]^{-1}$ which can be decomposed into the same component fermion propagators as above. Let us demonstrate this by calculating the 2-point functions $\Gamma^{(2)}_{\sigma \text{ or } \pi}$ of scalar σ and pseudoscalar π on this vacuum. For every flavor components σ^a and π^a, independently of $a = 0, 1, \cdots, n_{\mathrm{f}}^2 - 1$, they are given by

$$\begin{aligned}\Gamma^{(2)}_{\left\{ \sigma \atop \pi \right\}}(p^2) = &-\frac{\Lambda^2}{2g_{\mathrm{S}}^2} \\ &-\frac{N_{\mathrm{c}}}{2}\int\frac{d^4k}{i(2\pi)^4}\,\mathrm{tr}\left[\left\{ 1 \atop i\gamma_5 \right\}\frac{1}{m-[\not{k}]_\Lambda}\left\{ 1 \atop i\gamma_5 \right\}\frac{1}{m-[\not{k}+\not{p}]_\Lambda}\right]\ ,\end{aligned} \tag{3.12}$$

where $[\not{k}]_\Lambda$ is a shorthand notation for $\not{k}\left(1-k^2/\Lambda^2\right)$. The second term standing for the fermion one-loop diagram can be evaluated by using the B picture decomposition (2.18) for each propagator as follows:

$$\begin{aligned}&\text{second term} \\ &= -\frac{N_{\mathrm{c}}}{2}\sum_{i,j=0}^{2}\eta_i\eta_j\int\frac{d^nk}{i(2\pi)^n}\,\mathrm{tr}\left[\left\{ 1 \atop i\gamma_5 \right\}\frac{1}{m_i-\not{k}}\left\{ 1 \atop i\gamma_5 \right\}\frac{1}{m_j-(\not{k}+\not{p})}\right] \\ &= \frac{N_{\mathrm{c}}}{8\pi^2}\sum_{i,j=0}^{2}\eta_i\eta_j\Big[\big(m_i^2+m_j^2 \pm m_i m_j - \frac{p^2}{2}\big)\mathbf{L} - \big(\frac{m_i^2+m_j^2}{2} - \frac{p^2}{6}\big) \\ &\qquad\qquad + F_{\pm}(p^2; m_i, m_j)\Big]\ ,\end{aligned} \tag{3.13}$$

with

$$
\begin{aligned}
F_\pm(p^2;m_i,m_j) &\equiv \int_0^1 dx\,\left(2\overline{m}_{ij}^2 \pm m_i m_j - 3x(1-x)p^2\right)\ln\frac{\overline{m}_{ij}^2 - x(1-x)p^2}{\Lambda^2}\ , \\
\overline{m}_{ij}^2 &\equiv (1-x)m_i^2 + x m_j^2\ .
\end{aligned}
\tag{3.14}
$$

All the divergent terms proportional to $\mathbf{L}$ in (3.13) are seen to vanish if we use the identities $\sum_{j=0}^2 \eta_j = 0$ and $\sum_{j=0}^2 \eta_j m_j = 0$ in (2.23) and therefore we obtain

$$
\Gamma^{(2)}_{\left\{\sigma \atop \pi\right\}}(p^2) = -\frac{\Lambda^2}{2g_{\mathrm{S}}^2} + \frac{N_{\mathrm{c}}}{8\pi^2}\sum_{i,j=0}^{2}\eta_i\eta_j F_\pm(p^2;m_i,m_j)\ . \tag{3.15}
$$

In quite the same way we can calculate 2-point functions for the vector and axial vector mesons, $V_\mu \equiv (R_\mu + L_\mu)/2$ and $A_\mu \equiv (R_\mu - L_\mu)/2$: assuming $g_{\mathrm{V}}^2 = g_{\mathrm{V0}}^2$, they are given independently of the flavor by

$$
\begin{aligned}
&\Gamma^{(2)\,\mu\nu}_{\left\{V \atop A\right\}}(p^2) \\
&= g^{\mu\nu}\frac{\Lambda^2}{g_{\mathrm{V}}^2} - \frac{N_{\mathrm{c}}}{2}\sum_{i,j=0}^{2}\eta_i\eta_j\int\frac{d^4k}{i(2\pi)^4}\,\mathrm{tr}\left[\begin{Bmatrix}\gamma^\mu \\ \gamma^\mu\gamma_5\end{Bmatrix}\frac{1}{m_i - \not{k}}\begin{Bmatrix}\gamma^\nu \\ \gamma^\nu\gamma_5\end{Bmatrix}\frac{1}{m_j - (\not{k}+\not{p})}\right] \\
&= g^{\mu\nu}\left[\frac{\Lambda^2}{g_{\mathrm{V}}^2} - \frac{N_{\mathrm{c}}}{8\pi^2}\sum_{i,j}\eta_i\eta_j G_{\left\{V \atop A\right\}}(p^2;m_i,m_j)\right] \\
&\qquad\qquad + (g^{\mu\nu}p^2 - p^\mu p^\nu)\frac{N_{\mathrm{c}}}{8\pi^2}\sum_{i,j}\eta_i\eta_j H(p^2;m_i,m_j)\ ,
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
G_{\left\{V \atop A\right\}}(p^2;m_i,m_j) &\equiv \int_0^1 dx\,\left(\overline{m}_{ij}^2 \mp m_i m_j\right)\ln\frac{\overline{m}_{ij}^2 - x(1-x)p^2}{\Lambda^2}\ , \\
H(p^2;m_i,m_j) &\equiv \int_0^1 dx\,2x(1-x)\ln\frac{\overline{m}_{ij}^2 - x(1-x)p^2}{\Lambda^2}\ .
\end{aligned}
\tag{3.17}
$$

Although these Eqs.(3.15) and (3.16) are not so explicit expressions since the roots m_j are implicit functions of m and Λ, they give closed expressions for the 2-point

functions which allows straightforward numerical evaluations. (In particular, the parameter integrals over x in (3.14) and (3.17) can be carried out analytically and the functions $F_\pm$, $G_{\left\{ V \atop A \right\}}$ and H are given explicitly by elementary functions.)

We here note some points on these 2-point functions. Firstly the pion is indeed massless, $\Gamma_\pi^{(2)}(p^2=0)=0$, as it should be because of chiral symmetry. This can immediately be seen by putting $p=0$ directly in (3.12) and comparing it with the gap equation (3.6). [It is more complicated to see this if we use the expression (3.15), although being possible by the help of a formula $\sum_j \eta_j/(m_j+m_i) = 1/2m$ (m_i-independent) following from the relation $\sum_j \eta_j/(m_j-x) = -\Lambda^2/\prod_j(m_j-x)$.] Thus the pion 2-point function behaves as $\Gamma_\pi^{(2)}(p^2) = Z_\pi^{-1}p^2 + O(p^4)$ around $p^2=0$ and the coefficient Z_π^{-1} defines the (inverse of) wave-function renormalization factor of our π field as $\pi = Z_\pi^{\frac{1}{2}}\pi_{\rm r}$. Since there is no genuine Yukawa vertex correction in the $1/N_{\rm c}$ leading order, this wave-function renormalization factor also determines the Yukawa coupling $g_{\pi\bar\psi\psi}$ defined by $\mathcal{L}_{\rm int} = -g_{\pi\bar\psi\psi}\bar\psi i\gamma_5\pi_{\rm r}\psi$; namely, $g_{\pi\bar\psi\psi} = Z_\pi^{\frac{1}{2}}$. Evaluating the coefficient of p^2 in (3.15), we find

$$Z_\pi^{-1} = {g_{\pi\bar\psi\psi}}^{-2} = \frac{N_{\rm c}}{8\pi^2}\sum_{i,j=0}^{2}\eta_i\eta_j\frac{\partial F_-}{\partial p^2}(0;m_i,m_j)\ , \tag{3.18}$$

$$\begin{aligned}\frac{\partial F_-}{\partial p^2}(0;m_i,m_j) &= \left[\frac{-m_i^3(m_i+2m_j)\ln\frac{m_i^2}{\Lambda^2}}{2(m_i+m_j)^3(m_i-m_j)} + \big(i\leftrightarrow j\big)\right] + \frac{m_im_j}{2(m_i+m_j)^2} + \frac{1}{12} \\ &\qquad \left(\ \underset{m_i\to m_j}{\longrightarrow}\ -\frac{1}{2}\ln\frac{m_i^2}{\Lambda^2} - \frac{1}{6}\ \right)\ .\end{aligned} \tag{3.19}$$

Secondly, an interesting point in our definition of NJL model is that the scalar σ meson develops a pole *below* the two fermion threshold $2m_0$. Namely, the famous NJL mass relation $m_\sigma = 2m_0$ no longer holds here. We can find the σ meson mass numerically as a zero of $\Gamma_\sigma^{(2)}$ in (3.15) and show the result in Fig.1. Note that the NJL mass relation holds only in the limit $m_0/\Lambda\to 0$. [The approach to the

Fig.1. Meson mass squares m_b^2 vs fermion mass m_0. σ, V and A denote scalar, vector and axial-vector meson boundstates, respectively. (m_0/Λ is bounded by $1/\sqrt{3}$.)

Finally, as for the vector and axial-vector meson 2-point functions (3.16), we note that the one-loop contribution parts do not have gauge-invariant form $\propto$ $(g_{\mu\nu}p^2 - p_\mu p_\nu)$ even in the limit $m \to 0$ contrary to the true gauge fields. This is of

course because the apparent 'gauge-invariance' for those mesons is violated in the higher derivative terms in the present model. We have also plotted in Fig.1 the masses of the vector and axial-vector mesons determined as zeros of $\Gamma^{(2)}_{V,A}$ in (3.16) for the case $g_{\rm S}^2 = g_{\rm V}^2$ corresponding to QCD analogue model. It may be of some interest to note that, if we take $\Lambda = 1$GeV, 900MeV, 800MeV and $f_\pi = 93$MeV as inputs, the present NJL model gives the following values for the fermion mass m_0 and the fermion pair VEV:

$$\begin{aligned}
\Lambda = 1\,\mathrm{GeV}: &\quad m_0 = 250\,\mathrm{MeV}, \quad \left(- \left\langle \bar\psi\psi \right\rangle_{1\mathrm{GeV}} \right)^{1/3} = 253\,\mathrm{MeV} \\
\Lambda = 900\,\mathrm{MeV}: &\quad m_0 = 273\,\mathrm{MeV}, \quad \left(- \left\langle \bar\psi\psi \right\rangle_{1\mathrm{GeV}} \right)^{1/3} = 245\,\mathrm{MeV} \\
\Lambda = 800\,\mathrm{MeV}: &\quad m_0 = 315\,\mathrm{MeV}, \quad \left(- \left\langle \bar\psi\psi \right\rangle_{1\mathrm{GeV}} \right)^{1/3} = 237\,\mathrm{MeV}\ ,
\end{aligned} \tag{3.20}$$

where Eq.(4.7) below for f_π and Eqs.(3.18), (3.6) and (2.3) are used. Eq.(2.3) gives $-\left\langle \bar\psi\psi \right\rangle = (\Lambda^2/g_{\rm S}^2)m$, which we regard as the fermion pair VEV renormalized at the cutoff scale Λ. The fermion pair VEV's cited here are renormalized ones at $\mu = 1\,\mathrm{GeV}$ which we calculated using the following formula although the renormalization effects are small:

$$-\left\langle \bar\psi\psi \right\rangle_\mu = \left(\frac{\ln(\mu^2/\Lambda_{\rm QCD}^2)}{\ln(\Lambda^2/\Lambda_{\rm QCD}^2)} \right)^{4/9} \frac{\Lambda^2}{g_{\rm S}^2} m \tag{3.21}$$

with $\Lambda_{\rm QCD} \simeq 500\,\mathrm{MeV}$, where 4/9 is the anomalous dimension of $\bar\psi\psi$ in three flavored $SU(3)$ QCD. The empirical value for $\left(- \left\langle \bar\psi\psi \right\rangle_{1\mathrm{GeV}} \right)^{1/3}$ to be compared with these is $225 \pm 25\,\mathrm{MeV}$.

Fig.2. Feynman diagram contributing to f_π.

Note that the RHS of Eq.(4.2) should be evaluated at the mass shell $p^2 = 0$ in order to give the decay constant f_π. But the RHS itself, corresponding to the diagram in Fig.2, *is* defined even off the mass shell and hence the Eq.(4.2) may be regarded as defining an 'off-shell' decay constant $f_\pi(p^2)$ which coincides with the true f_π at $p^2 = 0$. With this understanding, let p^μ be off the mass shell for a while and multiply both sides of Eq.(4.2) by ip_μ. Then we can obtain

$$f_\pi(p^2)\,p^2 = \frac{g_{\pi\bar{\psi}\psi}N_\mathrm{c}}{2}\int\frac{d^4k}{i(2\pi)^4}\Biggl\{-\,\mathrm{tr}\left[\frac{1}{m-[\not{k}+\not{p}]_\Lambda}+\frac{1}{m-[\not{k}]_\Lambda}\right] + 2m\,\mathrm{tr}\left[\frac{1}{m-[\not{k}]_\Lambda}\gamma_5\frac{1}{m-[\not{k}+\not{p}]_\Lambda}\gamma_5\right]\Biggr\}\,, \tag{4.4}$$

by using an algebraic identity

$$\begin{aligned} p_\mu\mathcal{A}^\mu(k,k+p) &= [\not{k}+\not{p}]_\Lambda\gamma_5 + \gamma_5[\not{k}]_\Lambda \\ &= \bigl(m-[\not{k}]_\Lambda\bigr)\gamma_5 + \gamma_5\bigl(m-[\not{k}+\not{p}]_\Lambda\bigr) - 2m\gamma_5\ . \end{aligned} \tag{4.5}$$

Note that this identity (4.5) is just a higher derivative case version of the usual tree level Ward-Takahashi identity for the axial vector current. The two terms in the first trace in (4.4) become equal to each other by shifting the loop momentum (which is allowed now) and they give $-2m(\Lambda^2/g_\mathrm{S}^2N_\mathrm{c})\times(g_{\pi\bar{\psi}\psi}N_\mathrm{c}/2) = -mg_{\pi\bar{\psi}\psi}\Lambda^2/g_\mathrm{S}^2$ owing to the gap equation (3.6). So we find

$$f_\pi(p^2)\,p^2 = 2mg_{\pi\bar{\psi}\psi}\Biggl\{-\frac{\Lambda^2}{2g_\mathrm{S}^2}+\frac{N_\mathrm{c}}{2}\int\frac{d^4k}{i(2\pi)^4}\,\mathrm{tr}\left[\frac{1}{m-[\not{k}]_\Lambda}\gamma_5\frac{1}{m-[\not{k}+\not{p}]_\Lambda}\gamma_5\right]\Biggr\}\,. \tag{4.6}$$

But the quantity in the curly bracket here is just the same as the 2-point function $\Gamma_\pi^{(2)}(p^2)$ in Eq. (3.12) of pseudoscalar π, which we know behaves as $Z_\pi^{-1}p^2 + O(p^4)$ around $p^2 = 0$. Therefore, taking also account of the relation $g_{\pi\bar{\psi}\psi} = Z_\pi^{\frac{1}{2}}$, we find

$$f_\pi(p^2{=}0) = f_\pi = 2mg_{\pi\bar{\psi}\psi}Z_\pi^{-1} = 2mZ_\pi^{-\frac{1}{2}} = \frac{2m}{g_{\pi\bar{\psi}\psi}}\ . \tag{4.7}$$

Although we have derived this relation (4.7) by a rather explicit computation in the leading order in $1/N_\mathrm{c}$, it is in fact a direct consequence of the chiral

symmetry alone. Indeed, using canonical commutation in A (or B) picture and expressing π and σ fields in terms of fermion field, we can derive the chiral symmetry transformation law of the NG boson field π^b:

$$\delta(x^0)[ij^0_{5a}(x),\, \pi^b(0)] = \delta^4(x)\,\mathrm{tr}\left[\lambda^b\bigl(\lambda^a\sigma(0)+\sigma(0)\lambda^a\bigr)\right] \ . \tag{4.8}$$

Then, by using VEV of this equation and $\langle 0|\,\sigma(x)\,|0\rangle = m\mathbf{1}_{n_\mathrm{f}}$, we see that the current conservation $\partial_\mu j^\mu_{5a}(x)=0$ leads to

$$i\partial_\mu \langle 0|\,\mathrm{T}\, j^\mu_{5a}(x)\pi^b(0)\,|0\rangle = \delta(x^0)\,\langle 0|\,[ij^0_{5a}(x),\, \pi^b(0)]\,|0\rangle = 4m\delta^4(x)\delta^b_a \ . \tag{4.9}$$

This implies that the 2-point function $\langle 0|\,\mathrm{T}\, j^\mu_{5a}\pi^b\,|0\rangle$ in momentum space contains a massless pole and is given by $(p^\mu/p^2)4m\delta^b_a$. But, on the other hand, such massless pole comes from the NG boson intermediate state and hence the pole residue should be $2f_\pi p^\mu\cdot Z_\pi^{\frac{1}{2}}\delta^b_a$. We thus get an equality $2f_\pi Z_\pi^{1/2}=4m$, the same relation as the Eq.(4.7).

4.2. $\pi^0\to 2\gamma$ Amplitude

We now turn to calculate the amplitude of $\pi^0\to 2\gamma$ decay. The neutral pion π^0 corresponds to $Z_\pi^{\frac{1}{2}}\pi^3$ in the present notation and the photon γ couples to (e times) the vector current $j^\mu_a(x)$ with the flavor matrix λ^a replaced by the quark charge matrix $Q=\mathrm{diag}(2/3,-1/3,\cdots)$. The simplest way for calculating this amplitude is to use the B picture since the vector current is diagonal with respect to the index j of the component fermions ψ_j even in the B picture. The amplitude is calculated by a triangle diagram and is given by

$$\begin{aligned}
\mathcal{M}_{\pi_0\to\gamma\gamma} &= -e^2 g_{\pi\bar\psi\psi}N_\mathrm{c}\,\mathrm{tr}\,\bigl(QQ\frac{\lambda^3}{2}\bigr)\sum_{j=0}^{2}\eta_j T^{\mu\nu}(p,q;m_j)\ , \\
T^{\mu\nu}(p,q;m_j) &= \int\frac{d^nk}{i(2\pi)^n}\Biggl\{\mathrm{tr}\left[\frac{1}{m_j-(\not k-\not q)}\gamma^\nu\frac{1}{m_j-\not k}\gamma^\mu\frac{1}{m_j-(\not k+\not p)}(-i\gamma_5)\right] \\
&\qquad\qquad\qquad\qquad + \bigl[(p,\mu)\leftrightarrow(q,\nu)\bigr]\Biggr\}\ .
\end{aligned} \tag{4.10}$$

The evaluation of the integral of the amplitude $T^{\mu\nu}(p,q;m_j)$ is well known;[10] it gives

$$T^{\mu\nu}(p,q;m_j) = -\frac{1}{4\pi^2 m_j}\epsilon^{\mu\nu\alpha\beta}q_\alpha p_\beta \ , \tag{4.11}$$

so that we find

$$\begin{aligned}\mathcal{M}_{\pi_0\to\gamma\gamma} &= \frac{e^2 g_{\pi\bar\psi\psi}N_{\rm c}}{4\pi^2}\,\mathrm{tr}\,\big(QQ\frac{\lambda^3}{2}\big)\Big(\sum_{j=0}^{2}\frac{\eta_j}{m_j}\Big)\epsilon^{\mu\nu\alpha\beta}q_\alpha p_\beta \\ &= \frac{e^2 N_{\rm c}}{4\pi^2}\big(\frac{g_{\pi\bar\psi\psi}}{2m}\big)\,\mathrm{tr}(QQ\lambda^3)\epsilon^{\mu\nu\alpha\beta}q_\alpha p_\beta \ .\end{aligned} \tag{4.12}$$

Here we have used $\sum_j \eta_j/m_j = 1/m$ in Eq.(2.23). If $g_{\pi\bar\psi\psi}/2m = 1/f_\pi$, this exactly reproduces the well-known low energy theorem for $\pi^0 \to 2\gamma$. This is indeed the case because of the relation (4.7) as we confirmed in the above.

5. Simplified Treatment for the Case $\Lambda \gg m$

In some applications of NJL model (as in the top condensation scenario,[9] for instance), the cutoff Λ is much larger than the scale of the mass m and momenta which we discuss. In such cases we need not keep terms which are suppressed by $1/\Lambda^2$ in the effective action. Then we can take $\Lambda^2 \to \infty$ limit in all the terms not diverging in that limit. This simplifies the calculations considerably and the calculated results become very explicit ones containing no longer the implicit masses m_j.

We now show how to evaluate the effective action (3.1) in such cases. Only problem is the fermion one-loop term, which we expand as follows using the notation $[i\partial\!\!\!/]_\Lambda \equiv i\partial\!\!\!/(1+\partial\!\!\!/\partial\!\!\!//\Lambda^2)$:

$$\begin{aligned}\mathrm{TrLn}\big[\,[i\partial\!\!\!/]_\Lambda - \mathcal{M}\big] = \mathrm{Tr}\Bigg[&\mathrm{Ln}\big[\,[i\partial\!\!\!/]_\Lambda\big] - \frac{\mathcal{M}}{[i\partial\!\!\!/]_\Lambda} - \frac{1}{2}\Big(\frac{\mathcal{M}}{[i\partial\!\!\!/]_\Lambda}\Big)^2 - \frac{1}{3}\Big(\frac{\mathcal{M}}{[i\partial\!\!\!/]_\Lambda}\Big)^3 \\ &- \frac{1}{4}\Big(\frac{\mathcal{M}}{[i\partial\!\!\!/]_\Lambda}\Big)^4 - \Bigg(\sum_{n\ge5}\frac{1}{n}\Big(\frac{\mathcal{M}}{[i\partial\!\!\!/]_\Lambda}\Big)^n\Bigg)\Bigg] \ ,\end{aligned} \tag{5.1}$$

The first term in the RHS is an irrelevant constant and the second term vanishes as explained before. Now we consider the $\Lambda \to \infty$ limit of this quantity (5.1). Recall the propagator decomposition in A picture:

$$\frac{1}{[i\partial\!\!\!/]_\Lambda} = \frac{1}{i\partial\!\!\!/} - \frac{1}{2}\frac{1}{i\partial\!\!\!/ - \Lambda} - \frac{1}{2}\frac{1}{i\partial\!\!\!/ + \Lambda} \ , \tag{5.2}$$

and call the LHS regularized propagator, the first term $1/i\partial\!\!\!/$ in the RHS unregularized propagator and the second and third terms with masses $\pm\Lambda$ regulator propagators. In the last term $\mathrm{Tr}\big[-\sum_{n\geq 5}(1/n)(\mathcal{M}/i[\partial\!\!\!/]_\Lambda)^n\big]$ in Eq.(5.1), we substitute (5.2) for each propagator factor $1/i[i\partial\!\!\!/]_\Lambda$ and then it becomes a sum of various terms consisting of the unregularized and regulator propagators. But, from dimension counting, each term is ultraviolet convergent and therefore we can take the $\Lambda \to \infty$ limit directly for the loop integrands (*i.e.*, inside the functional trace Tr). Clearly then all the terms containing the regulator propagators at least once drop out and only the term $\mathrm{Tr}\big[-\sum_{n\geq 5}(1/n)(\mathcal{M}/i\partial\!\!\!/)^n\big]$ survives. If we add to the Tr operand of this term the quantity

$$\mathrm{Ln}\big[i\partial\!\!\!/\big] - \frac{\mathcal{M}}{i\partial\!\!\!/} - \frac{1}{2}\Big(\frac{\mathcal{M}}{i\partial\!\!\!/}\Big)^2 - \frac{1}{3}\Big(\frac{\mathcal{M}}{i\partial\!\!\!/}\Big)^3 - \frac{1}{4}\Big(\frac{\mathcal{M}}{i\partial\!\!\!/}\Big)^4,$$

then it reproduces the one-loop term $\mathrm{TrLn}\big[i\partial\!\!\!/ - \mathcal{M}\big]$ of the unregularized fermion. We thus find that

$$\begin{aligned}
&\mathrm{TrLn}\big[[i\partial\!\!\!/]_\Lambda - \mathcal{M}\big] = \mathrm{TrLn}\big[i\partial\!\!\!/ - \mathcal{M}\big] + \frac{i}{N_\mathrm{c}}\Gamma_\mathrm{count} + O\Big(\frac{1}{\Lambda^2}\Big) \ , \\
&\frac{i}{N_\mathrm{c}}\Gamma_\mathrm{count} \equiv -\bigg(\frac{1}{2}\Big(\frac{\mathcal{M}}{[i\partial\!\!\!/]_\Lambda}\Big)^2 + \frac{1}{3}\Big(\frac{\mathcal{M}}{[i\partial\!\!\!/]_\Lambda}\Big)^3 + \frac{1}{4}\Big(\frac{\mathcal{M}}{[i\partial\!\!\!/]_\Lambda}\Big)^4\bigg) \\
&\qquad\qquad + \bigg(\frac{1}{2}\Big(\frac{\mathcal{M}}{i\partial\!\!\!/}\Big)^2 + \frac{1}{3}\Big(\frac{\mathcal{M}}{i\partial\!\!\!/}\Big)^3 + \frac{1}{4}\Big(\frac{\mathcal{M}}{i\partial\!\!\!/}\Big)^4\bigg) \ ,
\end{aligned} \tag{5.3}$$

up to an irrelevant constant. [Precisely speaking, the last term and the second last term $\mathrm{Tr}\big[-(1/4)(\mathcal{M}/i[\partial\!\!\!/]_\Lambda)^4\big]$ in Eq.(5.1) separately have an infrared divergence and the discussion above is not so rigorous. But actually a more careful argument

can justify this expression (5.3) as is inferred from the fact that in this expression (5.3) the infrared divergence cancels between the terms $\mathrm{Tr}\big[-(1/4)(\mathcal{M}/i[\partial\!\!\!/]_\Lambda)^4\big]$ and $\mathrm{Tr}\big[(1/4)(\mathcal{M}/i\partial\!\!\!/)^4\big]$.]

The Tr operation (loop integral) of this expression (5.3) as a whole of course gives a convergent quantity. But the integral for each term separately is divergent. In practice, however, we can evaluate the integral for each term separately if we use dimensional regularization. The convergence as a whole implies that the pole terms $1/\bar{\epsilon}$ appearing from each term should cancel eventually. We can use this fact to check the calculations.

We can evaluate the 'counterterm' Γ_{count} defined in Eq.(5.3) in a closed form if we discard $O(1/\Lambda^2)$ terms since then only dimension 2 and 4 operators survive. By using Eq.(5.3), the effective action (3.1) reads:

$$
\begin{aligned}
\Gamma = \int d^4x \; [& -\frac{\Lambda^2}{2g_{\mathrm{S}}^2}\,\mathrm{tr}\left(\Sigma^\dagger\Sigma\right) + \frac{\Lambda^2}{2g_{\mathrm{V}}^2}\,\mathrm{tr}\left(R^\mu R_\mu + L^\mu L_\mu\right)] \\
& + \frac{N_{\mathrm{c}}}{i}\mathrm{TrLn}[\,i\partial\!\!\!/ - \mathcal{M}\,] + \Gamma_{\mathrm{count}} + O\big(\frac{1}{\Lambda^2}\big) \; .
\end{aligned}
\tag{5.4}
$$

Performing a straightforward (but a bit laborious) calculation we find that the counterterm Γ_{count} is given by

$$\Gamma_{count} = \Gamma^{(2)}_{count} + \Gamma^{(3)}_{count} + \Gamma^{(4)}_{count} \ ,$$

$$\begin{aligned}
&\Gamma^{(2)}_{count} \\
&= \frac{N_{\rm c}}{16\pi^2}\Bigg[2\Lambda^2\,{\rm tr}(\Sigma^\dagger\Sigma) + (\mathbf{L} - \frac{4}{3})\,{\rm tr}(\partial^\mu\Sigma^\dagger\partial_\mu\Sigma) - \frac{1}{2}\Lambda^2\,{\rm tr}(R^\mu R_\mu + L^\mu L_\mu) \\
&\qquad + \Big\{ -\frac{1}{6}(\mathbf{L} - \frac{7}{6})\,{\rm tr}\left[(\partial_\mu R_\nu - \partial_\nu R_\mu)^2\right] - \frac{1}{6}\,{\rm tr}\left[(\partial^\mu R_\mu)^2\right] + \big(R \to L\big)\Big\}\Bigg] \ , \\
&\Gamma^{(3)}_{count} \\
&= \frac{N_{\rm c}}{16\pi^2}\Bigg[-i(\mathbf{L} - \frac{7}{3})\,{\rm tr}\left[R^\mu(\Sigma^\dagger \overleftrightarrow{\partial}_\mu \Sigma) + L^\mu(\Sigma \overleftrightarrow{\partial}_\mu \Sigma^\dagger)\right] \\
&\qquad + \frac{i}{3}(\mathbf{L} - 2)\,{\rm tr}\left[(\partial^\mu R^\nu - \partial^\nu R^\mu)[R_\mu, R_\nu] + (R \to L)\right]\Bigg] \ , \\
&\Gamma^{(4)}_{count} \\
&= \frac{N_{\rm c}}{16\pi^2}\Bigg[-(\mathbf{L} - \frac{17}{6})\,{\rm tr}\left[(\Sigma^\dagger\Sigma)^2\right] + (\mathbf{L} - \frac{7}{3})\,{\rm tr}\Big(R^2\Sigma^\dagger\Sigma + L^2\Sigma\Sigma^\dagger\Big) \\
&\qquad - 2(\mathbf{L} - \frac{17}{6})\,{\rm tr}\Big(\Sigma R^\mu \Sigma^\dagger L_\mu\Big) \\
&\qquad + \Big\{ \frac{1}{6}(\mathbf{L} - \frac{29}{12})\,{\rm tr}\Big([R_\mu, R_\nu]^2\Big) - \frac{1}{12}\,{\rm tr}\left[(R^\mu R_\mu)^2\right] + \big(R \to L\big)\Big\}\Bigg] \ ,
\end{aligned} \tag{5.5}$$

where $\Gamma^{(n)}_{count}$ with $n = 2, 3, 4$ stand for the contributions from $(1/n){\rm TrLn}\big[-\big(\mathcal{M}/[i\partial\!\!\!/]_\Lambda\big)^n + \big(\mathcal{M}/i\partial\!\!\!/\big)^n\big]$, respectively, and $\mathbf{L}$ is the divergence factor $\ln\Lambda^2 - 1/\bar{\epsilon}$ introduced in (3.9). It should be noted that the divergent terms proportional to $\mathbf{L}$ in Γ_{count} are combined to yield just the following 'gauge covariant' form:

$$\frac{N_{\rm c}}{16\pi^2}\mathbf{L}\Big[{\rm tr}(D^\mu\Sigma^\dagger D_\mu\Sigma) - \frac{1}{6}\,{\rm tr}(F^{R\mu\nu}F^R_{\mu\nu} + F^{L\mu\nu}F^L_{\mu\nu})\Big] \tag{5.6}$$

with

$$\begin{aligned}
&D_\mu\Sigma \equiv \partial_\mu\Sigma - iL_\mu\Sigma + i\Sigma R_\mu \\
&F^X_{\mu\nu} \equiv \partial_\mu X_\nu - \partial_\nu X_\mu - i[X_\mu, X_\nu] \qquad (X = R, L) \ .
\end{aligned} \tag{5.7}$$

This 'gauge covariant' form for the divergent parts is a reflection of the fact that the

fermion one-loop term $\propto \mathrm{TrLn}\big[\,[i\partial\!\!\!/]_\Lambda - \mathcal{M}\big]$ formally reduces to a gauge invariant form $\mathrm{TrLn}\big[i\gamma^\mu(\partial_\mu - iR_\mu\mathcal{P}_\mathrm{R} - iL_\mu\mathcal{P}_\mathrm{L}) - \Sigma\mathcal{P}_\mathrm{R} - \Sigma^\dagger\mathcal{P}_\mathrm{L}\big]$ in the $\Lambda \to \infty$ limit as if R_μ and L_μ are the gauge fields of the local $U(n_\mathrm{f})_\mathrm{R} \times U(n_\mathrm{f})_\mathrm{L}$ symmetry. The finite parts, however, do not have such gauge invariance, of course.

This form of the effective action (5.4) with counterterm Γ_count is a very convenient one. The effect of the cutoff by higher derivative term is now isolated in the counterterm Γ_count and it is given explicitly in (5.5). All we have to do then is to calculate the simple fermion one-loop action $\mathrm{TrLn}[\,i\partial\!\!\!/ - \mathcal{M}\,]$ with no cutoff by using dimensional regularization. The divergences appearing there will be automatically canceled by the contributions from the counterterm Γ_count.

Let us demonstrate the simplicity of this way of calculations by taking an example — the effective potential $V(m) \equiv V\big(\Sigma = m\mathbf{1}_{n_\mathrm{f}}\big)$. The effective potential contribution from the unregularized fermion one-loop is easily evaluated as

$$
\begin{aligned}
\frac{1}{n_\mathrm{f}} V_{\text{unreg. 1-loop}}(m) &= -N_\mathrm{c} \int \frac{d^n p}{i(2\pi)^n} \,\mathrm{tr}\ln[m - p\!\!\!/] \\
&= -2N_\mathrm{c} \int \frac{d^n p}{i(2\pi)^n} \ln\left[m^2 - p^2\right] = -\frac{N_\mathrm{c}}{16\pi^2} m^4 \Big(\mathbf{L} - \frac{3}{2} + \ln\frac{m^2}{\Lambda^2}\Big)\ ,
\end{aligned}
\tag{5.8}
$$

while the contribution from the counterterm is found by substituting $\Sigma = m\mathbf{1}_{n_\mathrm{f}}$ into (5.5) to be

$$
\frac{1}{n_\mathrm{f}} V_\mathrm{count}(m) = -\frac{N_\mathrm{c}}{16\pi^2}\Big(2m^2\Lambda^2 - m^4\big(\mathbf{L} - \frac{17}{6}\big)\Big)\ . \tag{5.9}
$$

We see the divergence $\propto \mathbf{L}$ in (5.8) is actually canceled by this counterterm. Adding the tree contribution $n_\mathrm{f}(m^2\Lambda^2/2g_\mathrm{S}^2)$ also, we have

$$
\frac{1}{n_\mathrm{f}} V(m) = \frac{\Lambda^2}{2g_\mathrm{S}^2} m^2 - \frac{N_\mathrm{c}}{16\pi^2}\Big(2m^2\Lambda^2 + m^4\big(\ln\frac{m^2}{\Lambda^2} + \frac{4}{3}\big)\Big)\ . \tag{5.10}
$$

The validity of this expression is of course limited in the region $m^2/\Lambda^2 \ll 1$. We draw in Fig.3 pictures of this potential and the ‘exact’ potential (3.10) in Sect.3, for comparison.

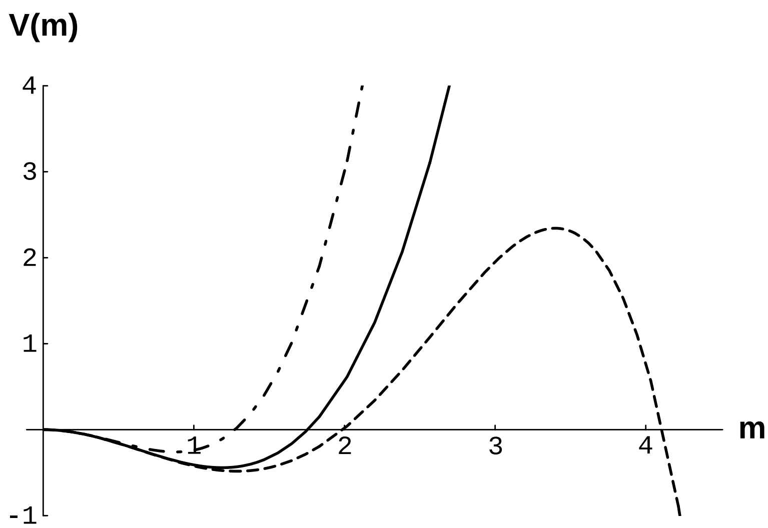

Fig.3. Effective potentials for the case $g_{\rm S} = 1.0193 g_{\rm S}^{\rm cr}$ above the critical coupling $g_{\rm S}^{\rm cr} = 2\pi/\sqrt{N_{\rm c}}$. Vertical and horizontal axes give scaled ones: $V(m) \times (16\pi^2 \cdot 10^3/N_{\rm c}\Lambda^4)$ and $m \times (10/\Lambda)$. Broken line denotes the approximate one (5.10), solid line the 'exact' one (3.10) and dotted-broken line the one (5.11) of the cutoff theory.

It may also be of some interest to compare this potential (5.10) with the usual effective potential obtained by the simple cutoff of the loop momentum which reads

$$\frac{1}{n_{\rm f}} V(m) = \frac{\Lambda^2}{2g_{\rm S}^2} m^2 - \frac{N_{\rm c}}{16\pi^2}\Big(m^2\Lambda^2 + \Lambda^4 \ln\big(1+\frac{m^2}{\Lambda^2}\big) - m^4 \ln\big(1+\frac{\Lambda^2}{m^2}\big)\Big) \ . \quad (5.11)$$

We have also drawn this potential function in Fig.3. [Its deviation from the present 'exact' one (3.10) in the large m region in Fig.3 reflects the difference in the meaning of the cutoff Λ in both theories.]

6. Conclusion

In this paper we have defined the NJL model by introducing higher derivative fermion kinetic term. We have clarified some basic aspects of the theory such as quantization, current operators *etc.*, and developed two calculation methods which make the evaluation of the diagrams in this higher derivative system no more difficult than in the usual first order derivative case.

We have emphasized that the present formulation of the NJL model suffers from no ambiguities in the loop momentum assignments and keeps the important chiral and gauge symmetries. We explicitly demonstrated this by calculating the $\pi_0 \to 2\gamma$ amplitude and confirming that the low energy theorem holds.

We restricted ourselves in this paper to the system with exact chiral symmetry. But there will be no problem in extending the present formalism to the cases where the fermions have explicit chiral symmetry breaking masses. When a fermion has an explicit mass M, we think it best to take the kinetic term in the form:

$$\bar{\psi}\big(i\partial\!\!\!/ - M\big)\left(1 + \frac{\partial\!\!\!/\partial\!\!\!/ + M^2}{\Lambda^2}\right)\psi \ . \tag{6.1}$$

Indeed, then the fermion really has mass M in the absence of interactions and satisfies correct normalization. The fermion momentum is effectively cut off as $|p^2 - M^2| \lesssim \Lambda^2$ around the mass shell. This form will be suitable also for describing heavy quarks which are much studied recently in connection with heavy quark symmetry.[4]

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APPENDIX Noether Current for a generic Higher Derivative System

We consider a generic system whose action contains arbitrary order derivatives:

$$S(\phi) = \int d^4x\, \mathcal{L}\big(\phi,\ \partial_\mu\phi,\ \partial_{\mu\nu}\phi,\ \partial_{\mu\nu\rho}\phi,\ \cdots\big)\ , \tag{A.1}$$

where ϕ stands for a collection of fields (whose index is suppressed) and we use abbreviations like

$$\begin{aligned} \partial_{\mu_1\mu_2\cdots\mu_n}\phi &\equiv \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_n}\phi\ , \\ \delta\mathcal{L}^{\,;\mu_1\mu_2\cdots\mu_n} &\equiv \frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1\mu_2\cdots\mu_n}\phi)}\Big|_{\text{weight 1}}\ . \end{aligned} \tag{A.2}$$

The suffix 'weight 1' in the latter means that we keep always the weight to be one irrespectively of whether the n indices $\mu_1, \mu_2, \cdots, \mu_n$ take the same values or not; namely, for the case $\mathcal{L} = a^{\mu\nu}\partial_{\mu\nu}\phi$, for instance, $\partial\mathcal{L}/\partial(\partial_{11}\phi) = a^{11}$ and $\partial\mathcal{L}/\partial(\partial_{12}\phi) = a^{12} + a^{21}$, but $\partial\mathcal{L}/\partial(\partial_{\mu\nu}\phi)\big|_{\text{weight 1}} = (a^{\mu\nu} + a^{\nu\mu})/2!$ always. The functional derivative of the action S with respect to ϕ is given by

$$\frac{\delta S}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu(\delta\mathcal{L}^{\,;\mu}) + \partial_{\mu\nu}(\delta\mathcal{L}^{\,;\mu\nu}) - \partial_{\mu\nu\rho}(\delta\mathcal{L}^{\,;\mu\nu\rho}) + \cdots\ , \tag{A.3}$$

and the Euler-Lagrange equation of motion is written as $\delta S/\delta\phi = 0$. If we perform

an infinitesimal transformation $\phi \to \phi + \delta\phi$, the Lagrangian $\mathcal{L}$ changes as

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + (\delta\mathcal{L}^{;\mu})\partial_\mu\delta\phi + (\delta\mathcal{L}^{;\mu\nu})\partial_{\mu\nu}\delta\phi + (\delta\mathcal{L}^{;\mu\nu\rho})\partial_{\mu\nu\rho}\delta\phi + \cdots \\ &= \frac{\delta S}{\delta\phi}\delta\phi + \big[(\delta\mathcal{L}^{;\mu})\partial_\mu\delta\phi + \partial_\mu(\delta\mathcal{L}^{;\mu})\cdot\delta\phi\big] + \big[(\delta\mathcal{L}^{;\mu\nu})\partial_{\mu\nu}\delta\phi - \partial_{\mu\nu}(\delta\mathcal{L}^{;\mu\nu})\cdot\delta\phi\big] \\ &\qquad + \big[(\delta\mathcal{L}^{;\mu\nu\rho})\partial_{\mu\nu\rho}\delta\phi + \partial_{\mu\nu\rho}(\delta\mathcal{L}^{;\mu\nu\rho})\cdot\delta\phi\big] + \cdots \ .\end{aligned} \tag{A.4}$$

To rewrite this, we introduce a generalized 'both-side' derivative defined by

$$\begin{aligned}F \overleftrightarrow{\partial}_{\mu_1\mu_2\cdots\mu_n} G \ \equiv\ & F\partial_{\mu_1\mu_2\cdots\mu_n}G - \partial_{\mu_1}F\cdot\partial_{\mu_2\cdots\mu_n}G \\ & + \partial_{\mu_1\mu_2}F\cdot\partial_{\mu_3\cdots\mu_n}G - \cdots + (-)^n\partial_{\mu_1\mu_2\cdots\mu_n}F\cdot G \ ,\end{aligned} \tag{A.5}$$

for arbitrary two functions F and G. This derivative is no longer symmetric under permutation of the indices but enjoys an identity

$$\partial_\mu\big[F^{\mu\alpha_1\cdots\alpha_n}\overleftrightarrow{\partial}_{\alpha_1\cdots\alpha_n}G\big] = F^{\mu\alpha_1\cdots\alpha_n}\partial_{\mu\alpha_1\cdots\alpha_n}G + (-)^n\partial_{\mu\alpha_1\cdots\alpha_n}F^{\mu\alpha_1\cdots\alpha_n}\cdot G \tag{A.6}$$

for functions $F^{\mu\alpha_1\cdots\alpha_n}$ totally symmetric with respect to the $n+1$ indices $\mu, \alpha_1, \cdots, \alpha_n$. Applying this identity we can write (A.4) as

$$\begin{aligned}\delta\mathcal{L} &= \frac{\delta S}{\delta\phi}\delta\phi + \partial_\mu\big[(\delta\mathcal{L}^{;\mu})\delta\phi\big] + \partial_\mu\big[(\delta\mathcal{L}^{;\mu\nu})\overleftrightarrow{\partial}_\nu\delta\phi\big] + \partial_\mu\big[(\delta\mathcal{L}^{;\mu\nu\rho})\overleftrightarrow{\partial}_{\nu\rho}\delta\phi\big] + \cdots \\ &= \frac{\delta S}{\delta\phi}\delta\phi + \partial_\mu\left[\sum_{n=0}^{\infty}(\delta\mathcal{L}^{;\mu\alpha_1\cdots\alpha_n})\overleftrightarrow{\partial}_{\alpha_1\cdots\alpha_n}\delta\phi\right] \ .\end{aligned} \tag{A.7}$$

If the transformation $\phi \to \phi + \delta\phi$ with $\delta\phi \equiv \varepsilon^a\hat{\delta}_a\phi$ (ε^a: x-independent transformation parameters) leaves the lagrangian invariant, $\delta\mathcal{L} = 0$, then we obtain from this an identity:

$$\begin{aligned}\partial_\mu j_a^\mu &= -\frac{\delta S}{\delta\phi}\hat{\delta}_a\phi \ , \\ j_a^\mu &\equiv \sum_{n=0}^{\infty}(\delta\mathcal{L}^{;\mu\alpha_1\cdots\alpha_n})\overleftrightarrow{\partial}_{\alpha_1\cdots\alpha_n}\hat{\delta}_a\phi \ .\end{aligned} \tag{A.8}$$

This j_a^μ gives a generalized Noether current and is seen to be conserved if the Euler-Lagrange equation $\delta S/\delta\phi = 0$ is satisfied.

For completeness we show here that this generalized Noether current coincides with the source current of the gauge field if it is introduced by the usual covariantization procedure replacing the derivative by a covariant one: $\partial_\mu \to D_\mu \equiv \partial_\mu - iA_\mu^a T^a$ for the case $\hat{\delta}_a\phi = iT^a\phi$ with a certain representation matrix T^a. Then noting

$$\begin{aligned}
&\partial_{\mu_1\mu_2\cdots\mu_n}\phi \ \rightarrow \\
&D_{\mu_1}D_{\mu_2}\cdots D_{\mu_n}\phi = \partial_{\mu_1\mu_2\cdots\mu_n}\phi - \sum_{k=1}^{n}\partial_{\mu_1\cdots\mu_{k-1}}\bigl(A^a_{\mu_k}\partial_{\mu_{k+1}\cdots\mu_n}\hat{\delta}_a\phi\bigr) + O(A^2)\ ,
\end{aligned} \tag{A.9}$$

we find the term linear in the gauge field in the covariantized lagrangian $\mathcal{L}_{\rm cov}(\phi, A)$ is given by

$$\begin{aligned}
\mathcal{L}_{\rm cov}(\phi, A)\big|_{A\text{-linear}} &= -\sum_{n=0}^{\infty}\sum_{k=0}^{n}(\delta\mathcal{L}^{\,;\mu\alpha_1\cdots\alpha_n})\partial_{\alpha_1\cdots\alpha_k}\bigl(A^a_\mu\partial_{\alpha_{k+1}\cdots\alpha_n}\hat{\delta}_a\phi\bigr) \\
&= -A^a_\mu\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(-)^k\partial_{\alpha_1\cdots\alpha_k}(\delta\mathcal{L}^{\,;\mu\alpha_1\cdots\alpha_n})\cdot\partial_{\alpha_{k+1}\cdots\alpha_n}\hat{\delta}_a\phi\right] + \bigl(\text{tot. der.}\bigr)\ ,
\end{aligned} \tag{A.10}$$

where we have performed a 'partial integration' in going to the second line and (tot. der.) denotes a total derivative term appearing then. We note that the quantity multiplied by A^a_μ in the first term is just identical with the above Noether current and so we have shown

$$-\frac{\delta S_{\rm cov}[\phi, A]}{\delta A^a_\mu}\bigg|_{A=0} = \ j^\mu_a\ . \tag{A.11}$$

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